Amplitude to phase noise conversion in electronic circuits

Edoardo Milotti*
Dipartimento di Fisica, Università di Trieste, Via Valerio 2, I-34127 Trieste, Italy
and INFN, Sezione di Trieste, Via Valerio 2, I-34127 Trieste, Italy
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The conversion mechanisms of amplitude noise to phase noise in high-precision oscillators are well established, but the converse is not true. In this paper phase to amplitude noise conversion is reviewed first, and a simple explanation is proposed for amplitude to phase noise conversion. [S1063-651X(97)03112-7]

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I. INTRODUCTION

A considerable body of literature deals with the conversion of noise to and from the amplitude and phase domains, with special emphasis on the topic of squeezed states [1]. Still other papers cover the topic of amplitude to phase noise conversion in semiconductor lasers [2]. Very general conversion mechanisms of phase noise to amplitude noise are also well established [3].

Nonetheless there is no known mechanism that may explain at the classical level the conversion of low-frequency $1/f$ amplitude noise to low-frequency phase noise: The works cited above — the one on squeezed states, for instance — do not help, since they deal essentially with quantum mechanical properties of the signal. This is unfortunate, because while the generation of low-frequency amplitude noise is fairly well understood [4] , the same is not true for phase noise, and a conversion mechanism would provide an easy way to explain the $1/f$ component of phase noise found in the highly stable oscillators used for precise timekeeping [5]: This low-frequency behavior of clocks is a severely limiting factor in many delicate measurements [6]. Here I briefly review the mathematics of phase to amplitude noise conversion, and I propose a simple mechanism that may account for amplitude to phase noise conversion.

II. PHASE TO AMPLITUDE NOISE CONVERSION

Spectral lines are broadened by phase noise, and this effect can be easily calculated as follows: Consider a sinusoidal signal with fixed amplitude A that starts at time t_1 and ends at time t_2, its Fourier transform (FT) is

$$F(\omega)=\int_{t_1}^{t_2} A\,\sin(\omega_0 t+\varphi)e^{-i\omega t}dt, \tag{1}$$

and if we let $\Delta t=t_2-t_1$, $t_0=(t_2+t_1)/2$, then we find

$$F(\omega)=\frac{A}{i}\left\{ e^{i\varphi}e^{i(\omega_0-\omega)t_0}\frac{\sin(\omega_0-\omega)\Delta t}{\omega_0-\omega} - e^{-i\varphi}e^{-i(\omega_0+\omega)t_0}\frac{\sin(\omega_0+\omega)\Delta t}{\omega_0+\omega}\right\}. \tag{2}$$

Now consider the signals produced by an oscillator such as a laser: each signal is coherent only over a limited time, since there are random events that introduce small phase shifts [7], and the whole signal of duration T can be split into a set of much shorter coherent pulses of individual duration Δt_j and centered at time t_j. A device like a laser acts as an oscillator with positive feedback and the (ideal) output signal has a constant amplitude, and because of the sum property of the FT's, the FT of the output signal is given by the sum of the FT's of the pulses, i.e., it is given by

$$F(\omega)=\frac{A}{i}\sum_j \left\{ e^{i\varphi_j}e^{i(\omega_0-\omega)t_j}\frac{\sin(\omega_0-\omega)\Delta t_j}{\omega_0-\omega} - e^{-i\varphi_j}e^{-i(\omega_0+\omega)t_j}\frac{\sin(\omega_0+\omega)\Delta t_j}{\omega_0+\omega}\right\}, \tag{3}$$

and the power spectral density (PSD) is proportional to

$$|F(\omega)|^2=A^2\sum_{j,k}\left[e^{i\varphi_j}e^{i(\omega_0-\omega)t_j}\frac{\sin(\omega_0-\omega)\Delta t_j}{\omega_0-\omega} - e^{-i\varphi_j}e^{-i(\omega_0+\omega)t_j}\frac{\sin(\omega_0+\omega)\Delta t_j}{\omega_0+\omega}\right]\left[e^{-i\varphi_k}e^{-i(\omega_0-\omega)t_k}\frac{\sin(\omega_0-\omega)\Delta t_k}{\omega_0-\omega} - e^{i\varphi_k}e^{i(\omega_0+\omega)t_k}\frac{\sin(\omega_0+\omega)\Delta t_k}{\omega_0+\omega}\right]$$

$$=\sum_j\left\{\frac{\sin^2(\omega_0-\omega)\Delta t_j}{(\omega_0-\omega)^2}+\frac{\sin^2(\omega_0+\omega)\Delta t_j}{(\omega_0+\omega)^2}\right\}+\sum_{j\neq k} e^{i(\varphi_j-\varphi_k)}e^{i(\omega_0-\omega)(t_j-t_k)}\frac{\sin(\omega_0-\omega)\Delta t_j}{\omega_0-\omega}\;\frac{\sin(\omega_0-\omega)\Delta t_k}{\omega_0-\omega}$$

*Electronic address: milotti@trieste.infn.it

$$+\sum_{j\neq k} e^{-i(\varphi_j-\varphi_k)}e^{-i(\omega_0+\omega)(t_j-t_k)}\frac{\sin(\omega_0+\omega)\Delta t_j}{\omega_0+\omega}\frac{\sin(\omega_0+\omega)\Delta t_k}{\omega_0+\omega}$$
$$-\sum_{j,k} e^{i(\varphi_j+\varphi_k)}e^{i(\omega_0-\omega)t_j}e^{i(\omega_0+\omega)t_k}\frac{\sin(\omega_0-\omega)\Delta t_j}{\omega_0-\omega}\frac{\sin(\omega_0+\omega)\Delta t_k}{\omega_0+\omega}$$
$$-\sum_{j,k} e^{-i(\varphi_j+\varphi_k)}e^{-i(\omega_0+\omega)t_j}e^{-i(\omega_0-\omega)t_k}\frac{\sin(\omega_0+\omega)\Delta t_j}{\omega_0+\omega}\frac{\sin(\omega_0-\omega)\Delta t_k}{\omega_0-\omega}. \quad (4)$$

The original order of the pulses in the signal gets lost in Eq. (4) so that, with a sufficiently long signal and with the phases performing a random walk so that at the end they are uniformly distributed in the interval $[0,2\pi)$, by averaging over the set of φ's one obtains

$$\langle|F(\omega)|^2\rangle = A^2\sum_j \left\{\frac{\sin^2(\omega_0-\omega)\Delta t_j}{(\omega_0-\omega)^2}+\frac{\sin^2(\omega_0+\omega)\Delta t_j}{(\omega_0+\omega)^2}\right\}. \quad (5)$$

Now, if we assume that the dephasing events have a Poisson statistics, so that their occurrence over time is given by an exponential distribution, the sum in Eq. (5) can be replaced by the integral

$$\langle|F(\omega)|^2\rangle = \frac{T}{\tau}A^2\int_0^\infty \frac{e^{-t/\tau}}{\tau}\left\{\frac{\sin^2(\omega_0-\omega)\tau}{(\omega_0-\omega)^2}+\frac{\sin^2(\omega_0+\omega)\tau}{(\omega_0+\omega)^2}\right\}dt$$
$$= TA^2\left(\frac{2\tau}{4(\omega_0-\omega)^2\tau^2+1}+\frac{2\tau}{4(\omega_0+\omega)^2\tau^2+1}\right) \quad (6)$$

where τ is the average time between two dephasing events, T/τ is the average number of pulses in the signal: Eventually the two-sided PSD (Φ^{PSD}) of the signal is given by

$$\Phi^{\mathrm{PSD}} = \lim_{T\to\infty}\frac{1}{T}\langle|F(\omega)|^2\rangle = A^2\left\{\frac{2\tau}{4(\omega_0-\omega)^2\tau^2+1}+\frac{2\tau}{4(\omega_0+\omega)^2\tau^2+1}\right\} \quad (7)$$

(in agreement with Ref. [3]). Therefore the PSD of the fixed amplitude signal with coherence time τ has a simple Lorentzian shape.

Now if we filter the signal, e.g., with a bandpass filter, the filtered output is amplitude modulated, i.e. there is a conversion of phase to amplitude noise. To see how this may happen, notice that the sudden phase change in the idealized signal implies a discontinuity in the function and this means that there cannot be any high-frequency cutoff in the PSD of the original signal, as in Eq. (7). If the signal is bandpass filtered the high-frequency components must necessarily disappear, and this leads to fluctuations near the discontinuity boundaries due to phase jumps (this is the usual Gibbs phenomenon, see, e.g., Ref. [8]): Figure 1 shows a practical example.

III. AMPLITUDE TO PHASE NOISE CONVERSION

We saw in Sec. II how a nonlinear mechanism such as bandpass filtering performs amplitude to phase noise conversion. The reverse is also possible if there is a corresponding nonlinear mechanism in the amplitude domain. Indeed, high-precision clocks count the number of periods of an oscillator with some sort of bistable thresholding device like a Schmitt trigger [9–11], which provides the proper nonlinearity.

Consider an amplitude modulated signal, like

$$f(t) = A[1+g(t)]\cos\omega_C t, \quad (8)$$

where ω_C is the carrier frequency, and $g(t)$ is the modulation envelope (usually band limited to a frequency band $[-W,W]$ such that $W \ll \omega_C$). Usually $|g(t)|<1$, but here we take $|g(t)|\ll 1$, so that $f(t)$ actually resembles the output of a stable oscillator. Obviously $f(t)$ has the same zeros as the carrier signal $\cos\omega_C t$, and this means that if one were able to use the zero crossings of $f(t)$ for counting the number of oscillator periods, then amplitude modulation would have no effect whatsoever on the counting precision. However, this is not usually the case, and we now consider two cases separately, i.e., the case of a nonzero threshold (with or without white noise added to the signal) and the case of a zero threshold (in the presence of some kind of white noise). In both instances we assume that the threshold device has hysteresis just as the above-mentioned Schmitt trigger: for positive threshold crossings (i.e., from below to above threshold) it is only the upper threshold of the Schmitt trigger that matters as long as the lower threshold is sufficiently distant to guarantee that there is no immediate retriggering because of the white noise added to the signal.

A. Nonzero threshold

As specified above, we now assume that counting occurs at the upward crossing of a positive threshold b (see Fig. 2): since the modulation envelope changes slowly, each half-period of $f(t)$ is well approximated by an arc of sinusoid with an amplitude equal to the instantaneous value of the modulation envelope, therefore the threshold crossing is phase delayed with respect to the zero crossing by an amount

$$\varphi(t) \simeq \omega_C \arcsin\frac{b}{A(t)}, \quad (9)$$

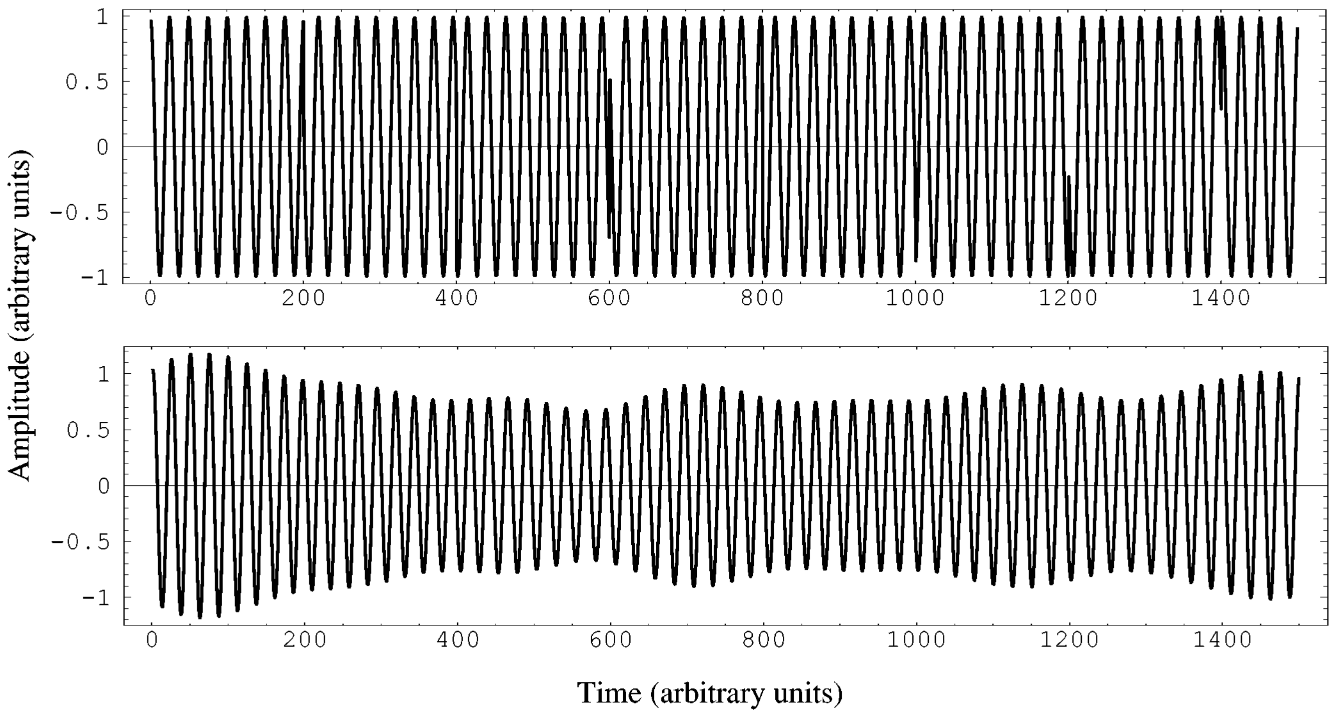

FIG. 1. A single signal composed of many pulses of equal length and with uniformly distributed phase jumps: These two rather unphysical assumptions have been made to emphasize the effect of bandpass filtering. The graph on the top shows the signal before filtering: all pulses have the same amplitude, and the phase jumps are clearly visible. The graph on the the bottom shows the signal after bandpass filtering: now it is smooth but the amplitude is no longer constant. The bandpass filter used in this example has $Q \simeq 16$.

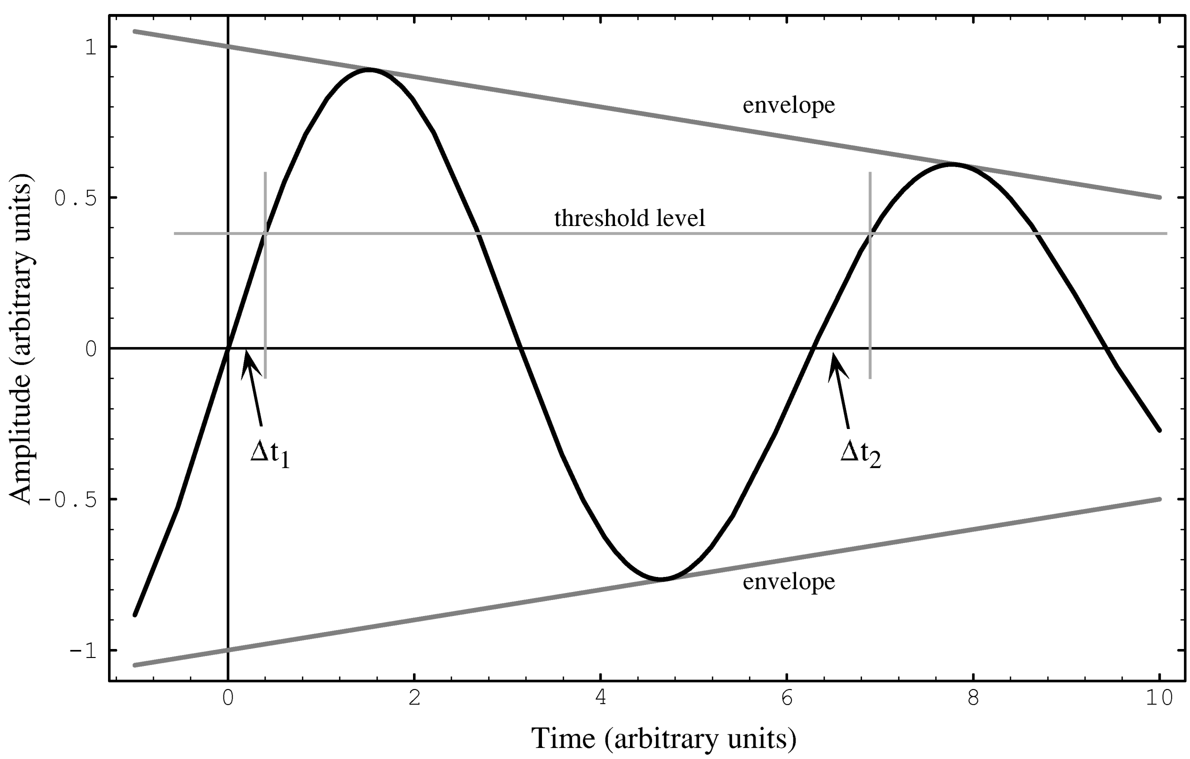

FIG. 2. This figure illustrates what happens when a slow amplitude modulation is present in a thresholding circuit: each half-period of the amplitude modulated wave form is approximately sinusoidal and half periods with a smaller amplitude lead to greater phase delays for a given threshold. Since we approximate each half period with a sinusoid, we can easily estimate the delays Δt_1 and Δt_2 (as explained in the main text).

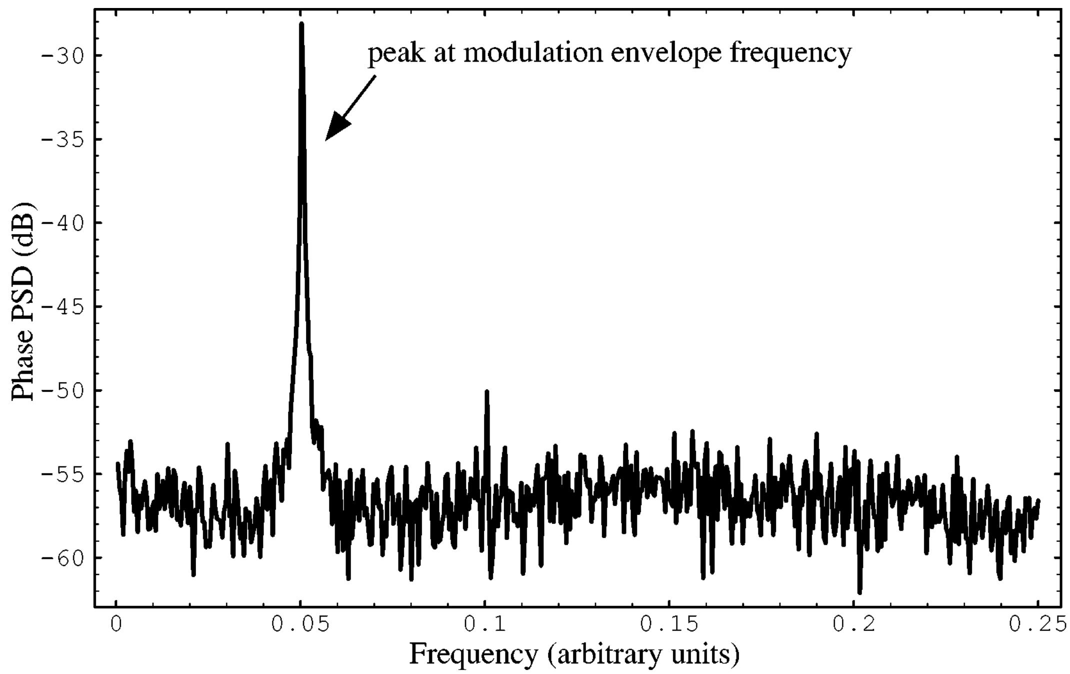

FIG. 3. This figure shows a the phase PSD obtained in a numerical simulation with a nonzero threshold. Here the wave form is not modulated by a noise processes but by a fixed frequency term with a frequency ten times lower than the carrier frequency; Gaussian white noise is also present in the original wave form. The PSD is the average of eight spectral densities, and each spectral density has been obtained from 1024 samples.

where $A(t)=A+\Delta A(t)=A[1+g(t)]$ is the instantaneous amplitude of the carrier signal (by hypothesis, the modulation envelope changes very slowly with respect to the carrier signal, and therefore this ''instantaneous'' value must be understood as ''average magnitude of the modulation envelope over one half-period of the carrier''). Then, if

$$\varphi_0=\omega_C \arcsin\frac{b}{A} \tag{10}$$

is the phase delay for a constant amplitude signal, we find that for a modulated signal

$$\begin{aligned}\varphi(t)=\varphi_0+\Delta\varphi(t)&\simeq\omega_C \arcsin\frac{b}{A+\Delta A(t)}\\ &\simeq\omega_C\left(\arcsin\frac{b}{A}-\frac{b}{A\sqrt{A^2-b^2}}\Delta A(t)\right),\end{aligned} \tag{11}$$

i.e.,

$$\Delta\varphi(t)\simeq-\omega_C\frac{b}{A\sqrt{A^2-b^2}}\Delta A(t)=-\omega_C\frac{b}{\sqrt{A^2-b^2}}g(t), \tag{12}$$

so that

$$\Phi^{\mathrm{PSD}}(\varphi(t))\simeq\frac{\omega_C^2 b^2}{A^2(A^2-b^2)}\Phi^{\mathrm{PSD}}(A(t)), \tag{13}$$

and therefore any amplitude noise is converted to a phase noise by the thresholding device, and the phase PSD is proportional to the amplitude PSD. Figure 3 shows the result of a numerical simulation.

B. Zero threshold with white noise

It is clear from Eq. (13) that, as the threshold value b approaches 0, the previous mechanism fails, and is unable to account for the conversion of amplitude to phase noise. However, noise in oscillators usually contains white noise as well as low-frequency colored noise, and this combines with the previous result to give once again a phase PSD proportional to the amplitude PSD.

To see how this may happen, consider a uniformly distributed white noise, i.e., a noise process $n(t)$ that has a uniform distribution with zero mean and width s, and such that $\langle n(t)n(t')\rangle=(s^2/12)\delta(t-t')$ (we deviate here from the usual assumption of Gaussian noise, because in this case uniformly distributed white noise is so much easier to treat). Now assume that the noisy signal is sampled at discrete intervals: if the noise standard deviation is much smaller than the signal amplitude, the sinusoidal signal can be approximated by a linear function with slope a in the zero crossing region, and each signal sample has a mean value $\overline{x}=at'$, where the time t' is measured relative to the zero crossing, as in Fig. 4. Since the noise has a uniform distribution, the zero crossing may take place in a limited time span $-s/2a<t'<s/2a$, as shown in Fig. 4. Assume also that the signal amplitude x is sampled at N uniformly spaced times $t_k=-s/2a+ks/na$, with $k=1,\ldots,n$ (the sampling sequence starts exactly at the beginning of the useful time span, but this special hypothesis is not very relevant if we let $n\gg 1$ at the end). Then $\overline{x}_k=at'_k=-s/2+ks/n$, and the probability that the sampled amplitude is above threshold is

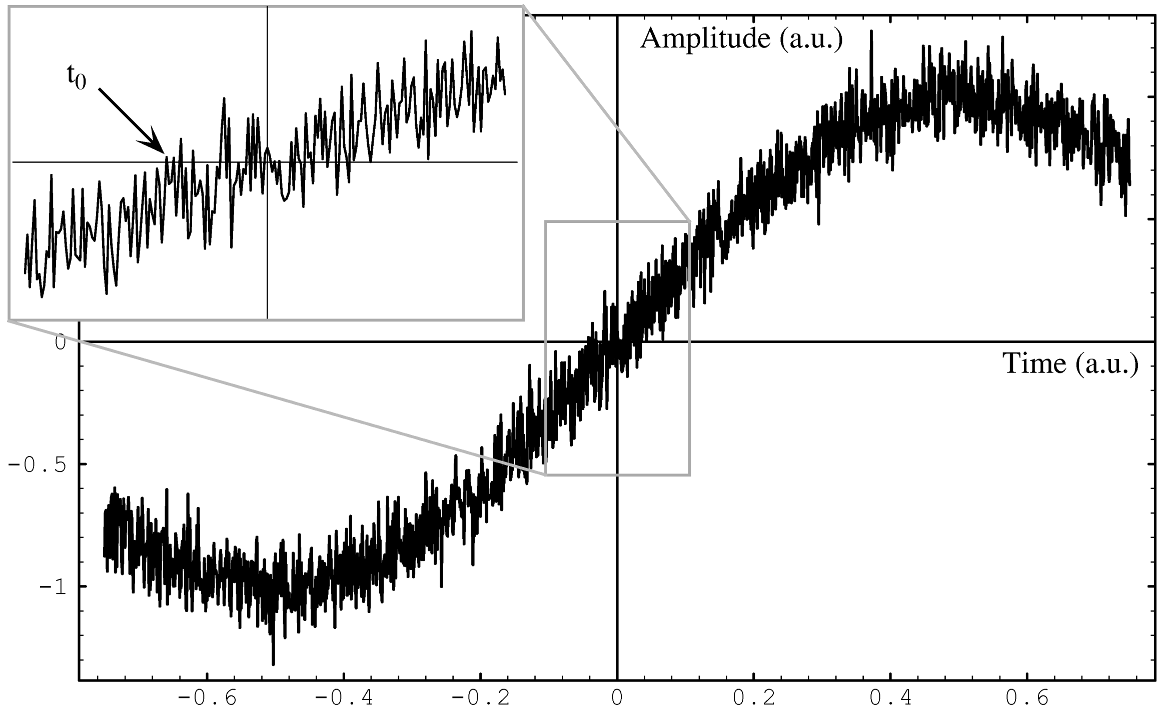

FIG. 4. This figure illustrates what happens to threshold crossing in the case of a sampled signal with uniformly distributed white noise: because of noise, triggering occurs earlier than expected (in this example at time t_0), when the signal is — on average — different from zero.

$p_k=(s/2+\overline{x_k})/s=k/n$, while the probability that the sampled amplitude is below zero is $q_k=1-p_k=1-k/n$. The probability that the threshold crossing takes place *exactly* at the kth sample is given by the probability that the $k-1$ previous samples are all below threshold, and that the kth sample is above threshold, i.e., by the expression

$$P_k=\left[\left(\prod_{j=0}^{j=k-1} q_j\right)\right]p_k=\left[\prod_{j=0}^{j=k-1}\left(1-\frac{j}{n}\right)\right]\frac{k}{n}=\frac{k}{n^{k+1}}\frac{n!}{(n-k)!}. \tag{14}$$

This expression can be expanded using the Stirling approximation:

$$\ln P_k \approx \ln k-(k+1)\ln n+(n\ \ln n-n) \\ -[(n-k)\ln(n-k)-(n-k)] \tag{15}$$

$$=\ln k-(k+1)\ln n+n\ \ln n-(n-k)\ln(n-k)-k, \tag{16}$$

and therefore if we treat k as if it were a continuous variable, we find that P_k peaks when

$$\frac{d\ \ln P_k}{dk}=\frac{1}{k}-\ln n+\ln(n-k)=0,$$

i.e., when

$$\exp\frac{1}{k}=\frac{n}{n-k}=\frac{1}{1-k/n}.$$

It is easy to see that, as n grows, the maximum of this probability moves to lower and lower values of k/n, i.e., as the number of samples increases, it is more and more probable to find the threshold crossing at the very beginning of the sampling interval $-s/2a<t'<s/2a$. This agrees qualitatively with the result of Rice on the threshold crossing rate of a noise process [12], which leads to a divergent crossing rate for noises with power spectra proportional to ω^{-k} with $k\leqslant 4$ — which means that the average waiting time for threshold crossing is negligibly small for such noise processes and for white noise in particular.

Actually, the PSD of any real noise process falls to zero at high frequency, so that the argument eventually fails, but nevertheless it indicates that on average the zero crossing happens at a time — in the example $t'\simeq -s/2a$ — when the oscillator signal is not zero: This means that there is a phase advance

$$\varphi=\omega_C t'\simeq-\omega_C\frac{s}{2a}, \tag{17}$$

and, since $a=A\omega_C$, then

$$\varphi(t)\simeq-\frac{s}{2A(t)}. \tag{18}$$

If

$$\varphi_0\simeq-\frac{s}{2A} \tag{19}$$

is the phase advance for a fixed amplitude signal, then

$$\varphi(t)=\varphi_0+\Delta\varphi(t)\simeq-\frac{s}{2[A+\Delta A(t)]} \tag{20}$$

and

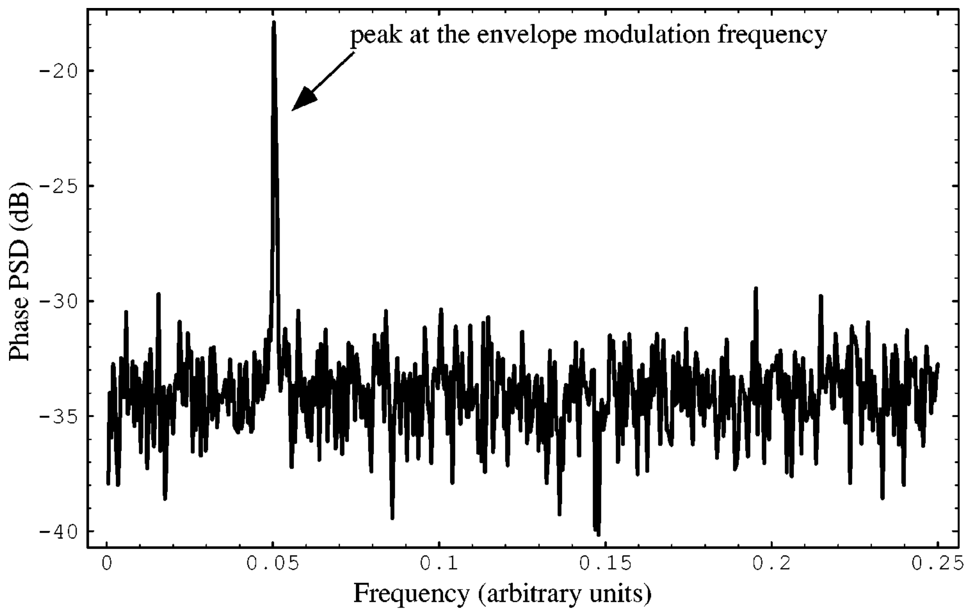

FIG. 5. This figure shows a the phase PSD obtained in a numerical simulation with threshold set at zero. Here the wave form is not modulated by a noise processes but by a fixed frequency term with a frequency ten times lower than the carrier frequency; Gaussian white noise is also present in the original wave form. The PSD is the average of eight spectral densities, and each spectral density has been obtained from 1024 samples.

$$\Delta\varphi(t) \simeq \frac{s}{2A^2}\Delta A(t) = \frac{s}{2A}g(t). \tag{21}$$

Therefore we find, as in the previous section, that the phase PSD is proportional to the amplitude PSD, i.e.,

$$\Phi^{\mathrm{PSD}}(\Delta\varphi(t)) \simeq \frac{s^2}{4A^4}\Phi^{\mathrm{PSD}}(\Delta A(t)). \tag{22}$$

The argument given above is not a real proof, but it shows that it is plausible to expect an amplitude to phase noise conversion even when the threshold is set at zero amplitude: This conclusion is also supported by numerical simulations. Figure 5 shows the PSD of the phase of a sinusoidal signal modulated at a fixed frequency plus a white Gaussian noise: the peak corresponds to the frequency of the modulation envelope.

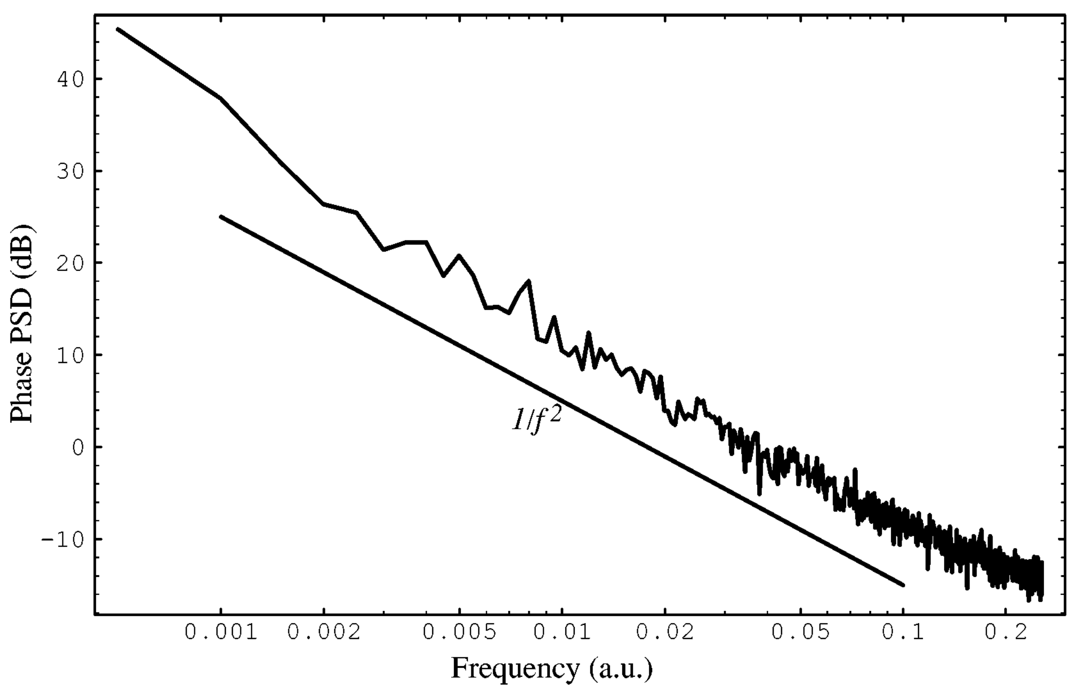

FIG. 6. This is what happens when the lower threshold is set incorrectly: the triggering circuit does not only trigger at the correct times but at other times in between. The clock rate is thus higher, and there are additional and rather large phase fluctuations. These phase fluctuations appear over many length scales, and thus the resulting phase PSD follows a power law. It turns out in the numerical simulations that a Gaussian white noise leads to a $1/f^2$ PSD.

C. Wrong setting of lower threshold

It is worthwhile to note that there is an additional effect that shows up in the simulations when the lower threshold is set incorrectly, i.e., when it happens that the bistable circuit often retriggers almost immediately after one good trigger. In these cases, the average crossing frequency is higher than the real crossing frequency, and the crossing times show large fluctuations with respect to the average. The numerical simulations performed with a Gaussian white noise show that the PSD of these fluctuations follows a power law, and in particular that $\Phi^{\mathrm{PSD}}(\varphi(t)) \propto \omega^{-2}$ (see Fig. 6).

D. Comments on the numerical simulations

A few caveats are in order when dealing with the numerical simulations. First it is not reasonable to expect a linear behavior such as that in Eqs. (13) and (22) for large modulation signals: This means that rather lengthy simulations are required to bring the tiny peaks out of noise. The simulations must necessarily use sampled signals, and because of the limited numerical accuracy of any computer this is bound to produce beats between the sampling frequency and the modulation frequency: one of the many solutions that help overcome this problem is a very small randomization of the sampling time. However, even if the simulations involve a sampling mechanism they are — after all — quite realistic: a real circuit like the commercial TTL Schmitt trigger 7414 has a finite response time (in this case for most circuits on the market it is of the order of 10^{-8} s) and this is the equivalent of a sampling time.

IV. SUMMARY AND CONCLUDING REMARKS

In this paper I have described a mechanism that may account for the appearance of low-frequency phase noise in high-precision oscillators. Low-frequency noise is the main limiting factor to high-precision timekeeping, and thus an understanding of the origin of this noise is obviously important for many physics applications of these fundamental measurement techniques [6,13]. The basic idea is that the nonlinear threshold device used to count cycles, and the ever present white noise, are the culprits of amplitude to phase noise conversion, in a way that is reminiscent of the increase of signal to noise ratio in stochastic resonance [14,15]. An accurate choice of threshold and a minimization of white noise may thus help reduce low-frequency noise.

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